Secant-hyperbolic instability in a reaction-diffusion system

Shrabani Sen and Deb Shankar Ray*
Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700 032, India
(Received 1 March 2009; published 8 May 2009)

We have shown that the nonlinearity of the chemical reaction may induce instability on a homogenous stable steady state of a one-component reaction-diffusion system characterized by a cubic polynomial source term. This results in a growth of an asymptotically approaching inhomogeneous spatial profile of secant-hyperbolic form reminiscent of a solitary wave.

DOI: 10.1103/PhysRevE.79.057101 PACS number(s): 82.40.Ck, 05.45.Yv, 47.54.−r

Reaction-diffusion systems [1–4] are ubiquitous in diverse areas of physical, chemical, and biological sciences. They provide useful description for a class of self-organization phenomena under far from equilibrium condition. The examples include among others the classical problem of propagation of flame [4,5], nerve impulse [6–8], chemical wave front [3,9,10], formation [3,11] of stationary and nonstationary spatial patterns, targets, and spirals. A major basis of theoretical analysis of these phenomena primarily rests on the linear stability of the homogenous steady state of the dynamical system under infinitesimal spatiotemporal perturbation [12]. However, when the perturbation is finite it is necessary to keep track of the spatiotemporal evolution of nonlinear terms. A number of instances [13] in this context are worth mentioning. For example, Lyapunov exponents have been generalized [13] by taking care of the nonlinearity due to quadratic or cubic terms. Finite perturbations have also been employed to introduce [14] a scale-dependent Lyapunov exponent for measuring the degree of chaoticity and also to coupled map lattices [15] for analyzing the nonlinear contribution to the velocity of propagation. Nonlinear analysis plays a significant role in determining the stability threshold in noise-induced pattern formation [16–20], understanding spatiotemporal instability due to finite relaxation time of the diffusive flux [21] and also in the Galerkin scheme of analysis of the nature of spatial patterns [22] and their cross over under variation in parameter space and in some other cases.

The focal theme of the present Brief Report is to understand how the spatiotemporal evolution of the nonlinear terms makes its presence felt in questions concerning stability of the steady states of a reaction-diffusion system. To this end we begin by noting that diffusion always tends to homogenize a one-component system. This traditional wisdom owes its basis to linear stability analysis [3] which shows that infinitesimal spatiotemporal perturbation on a homogenous steady state cannot give rise to instability. For more than one component system, however, diffusion may induce Turing instability [3,23]due to an interplay between short-range activation and long-range diffusion. The question is can instability be generated in a one-component reaction-diffusion system when the homogeneous steady state is perturbed by a finite spatiotemporal perturbation. Since the nonlinear response of the system crucially depends on the higher-order derivatives of the source function evaluated at the steady state in question, the dynamics of finite perturbation is expected to be generically different from what is normally obtained from the corresponding linear stability analysis. We address this issue with the help of a cubic polynomial source term which has served as a paradigm for many reaction-diffusion systems over the last several decades [4–10]. It has been shown that nonlinearity of the reaction may induce an instability on a homogenous stable steady state giving rise to a stationary inhomogeneous spatial pattern of sec-hyperbolic form in the long-time limit. Our numerical simulation depicting the temporal development of the perturbation corroborates our analytic results on the reaction-induced instability and the resulting inhomogeneous pattern.

To start with we consider a reaction-diffusion system which describes the dynamics of field variable $u(x,t)$, a function of space (x) and time (t),

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u). \tag{1}$$

D is the diffusion coefficient for the field variable. $f(u)$ is the source function derivable from an appropriate kinetic scheme for chemical reaction or otherwise. The homogenous steady states of the dynamical system are the fixed points u_0 defined by

$$f(u_0) = 0. \tag{2}$$

The spatiotemporal perturbation $\delta u(x,t)$ on a homogenous steady state u_0 is given by

$$u(x,t) = u_0 + \delta u(x,t). \tag{3}$$

The stability of the homogenous steady state against a finite perturbation, in general, is determined by the derivatives of the source term $f(u)$ evaluated at the steady state, i.e., $f'(u_0)$, $f''(u_0)$, and so on. The usual scheme of linearization is untenable. For the present problem we look for a class of polynomials of cubic variety. Our choice is guided by the following considerations. First the cubic polynomials are well known in the realm of autocatalytic chemical reactions [3,9,10]. They were also employed by Zeldovich *et al.* [5] and Scott [4] several decades ago for calculating the velocity of flame propagation. Cubic polynomial serves as a standard source term in the Hodgkin-Huxley model [6] and its variant

*pcdsr@mahendra.iacs.res.in

Fitzugh-Nagumo model [7] in impulse propagation along the active nerve fiber. Second, our choice of cubic polynomial puts a restriction on the number of higher-order derivative terms determining the dynamics of perturbation $\delta u(x,t)$. Thus the derivatives higher than the third order are all zero. This decisive mathematical advantage renders our analysis exactly solvable under stationary condition.

To proceed further we assume first that the homogenous steady state u_0 is stable, i.e.,

$$f'(u_0) < 0, \tag{4}$$

and furthermore the explicit form of $f(u)$ is such that

$$f''(u_0) = 0. \tag{5}$$

Based on these considerations we are led to the following equations for $\delta u(x,t)$ acting on the homogenous stable state:

$$\frac{\partial}{\partial t}\delta u = D\frac{\partial^2}{\partial x^2}\delta u + f'(u_0)\delta u + \frac{f'''}{3!}\delta u^3. \tag{6}$$

In what follows we show that it is possible to realize an inhomogeneous stationary state as a result of instability under the action of finite perturbation. To this end we note that for such a stationary state we must have

$$\frac{\partial}{\partial t}\delta u = 0. \tag{7}$$

Equation (6) therefore reduces to

$$D\frac{\partial^2}{\partial x^2}\delta u_s = -f'\delta u_s - \frac{f'''}{3!}\delta u_s^3, \tag{8}$$

where δu_s is the steady-state value of the perturbation. Abbreviating δu_s as $\xi(x)$ and

$$-\frac{f'(u_0)}{D} = \alpha \quad \text{and} \quad \frac{f'''(u_0)}{3\,!\,D} = \gamma, \tag{9}$$

Eq. (8) takes the form

$$\frac{\partial^2\xi}{\partial x^2} = \alpha\xi - \gamma\xi^3. \tag{10}$$

Multiplying Eq. (10) by $2\xi'$ and on integration we obtain

$$\left(\frac{d\xi}{dx}\right)^2 = \alpha\xi^2 - \frac{\gamma}{2}\xi^4 + A. \tag{11}$$

Here A is an integration constant which can be determined from the sum and product of the two roots of the biquadratic equation $F(\xi)=0$, where

$$A + \alpha\xi^2 - \frac{\gamma}{2}\xi^4 \equiv F(\xi). \tag{12}$$

Expressing

$$F(\xi) = (\alpha_1 - \alpha_2\xi^2)(\beta_1 - \beta_2\xi^2), \tag{13}$$

so that

$$A = \alpha_1\beta_1,$$

$$\alpha = -(\alpha_1\beta_2 + \alpha_2\beta_1),$$

$$\frac{\gamma}{2} = -\alpha_2\beta_2, \tag{14}$$

we obtain

$$x = \int_0^{\xi}\frac{d\xi}{\sqrt{A + \alpha\xi^2 - \frac{\gamma}{2}\xi^4}}. \tag{15}$$

By virtue of Eq. (4) and first of the relations in Eq. (9) we have $\alpha > 0$. Assume furthermore that $\gamma > 0$. Making use of Eqs. (12) and (13) we put Eq. (15) in the following form:

$$x = \int_0^{\xi}\frac{d\xi}{\sqrt{\alpha_1}\sqrt{\beta_1}\sqrt{\left(1-\frac{\alpha_2}{\alpha_1}\xi^2\right)}\sqrt{\left(1-\frac{\beta_2}{\beta_1}\xi^2\right)}}. \tag{16}$$

To have Eq. (16) in a more convenient form we let $\sqrt{\frac{\alpha_2}{\alpha_1}}\xi = u$ and $\sqrt{\frac{\beta_2}{\beta_1}}\xi = \kappa u$ so that $\kappa = \sqrt{\frac{\alpha_1\beta_2}{\beta_1\alpha_2}}$ and with $\sqrt{\alpha_2\beta_1} = \sigma$, we obtain

$$\sigma x = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-\kappa^2u^2)}}, \tag{17}$$

which is a well-known elliptic integral of first kind. The final solution can be expressed in terms of the Jacobian elliptic function

$$u = sn(\sigma x, \kappa). \tag{18}$$

The form of asymptotic perturbation is given by

$$\delta u_s(x) = \sqrt{\frac{\alpha_1}{\alpha_2}} sn(\sigma x, \kappa). \tag{19}$$

For a particular case of importance where $A=0$ with $\alpha > 0$ and $\gamma > 0$, we have from Eq. (11)

$$\sqrt{\alpha}x = \int_0^{\xi}\left[\xi^2\left(1 - \frac{\gamma}{2\alpha}\xi^2\right)\right]^{-1/2} d\xi. \tag{20}$$

Substituting $(\frac{\gamma}{2\alpha})^{1/2}\xi = \text{sech}\,\theta$ and on integration and rearrangement, Eq. (20) yields

$$\xi(x) = \left(\frac{2\alpha}{\gamma}\right)^{1/2} \text{sech}\sqrt{\alpha}x. \tag{21}$$

Hence the asymptotic form of finite perturbation in the long-time limit is given by

$$\delta u_s(x) = \left[\frac{12(-f'(u_0))}{f'''(u_0)}\right]^{1/2} \text{sech}\sqrt{\left(\frac{-f'(u_0)}{D}\right)}x. \tag{22}$$

Equation (22) is a key result of this Brief Report. It shows that a finite spatiotemporal perturbation destabilizes a homogenous stable steady state initiating the growth of an inhomogeneous distribution of sech form in the stationary state. A close look at this expression suggests the presence of a third derivative of the source term as a hallmark of nonlinear excitation of the system. This instability in a one-component system is thus essentially reaction induced in contrast to the diffusion-driven Turing instability in two-component reaction-diffusion systems. While Turing instability is amenable to a linear stability analysis, the generic origin of the reaction-induced instability lies in the nonlinearity of the source term and is independent of the length scale of system. We mention, in passing, that the solitary wavelike stationary solution (22) is reminiscent of the solitary wave solution [2,4,24,25] of classical integrable systems such as Kortewegde Vries equation (KdV), Sine-Gordon, and also in the context of light propagation in dispersive nonlinear quasiperiodic stratified media and in three-wave mixing phenomena.

In order to illustrate the above theoretical scheme we now assume a typical form of cubic source function as follows:

$$f(u) = u(u-1)(u+1). \tag{23}$$

The homogeneous steady states are given by $u_0=0$, $+1$, and -1. $u_0=0$ is the linearly stable state. $f(u)$ satisfies all the basic conditions for the reaction-driven instability. We thus have

$$f'(0) = -1, \quad f''(0) = 0, \quad \text{and} \quad f'''(0) = 6. \tag{24}$$

At this point it is also pertinent to note that although the form of finite perturbation in the long-time limit attains a stationary inhomogeneous distribution, the analysis shed no light on the approach toward this state. In order to address the issue we explore the temporal development of the perturbation by carrying out numerical simulations of the reaction-diffusion system with source function (23). To this end computations were performed using the explicit Euler method on an one-dimensional grid of 100 cells with $\Delta x=0.5$ and time step $\Delta t=0.1$, under zero concentration boundary condition. The value of the diffusion coefficient is set as $D=0.5$. The simulations were started with spatially random perturbations around the chosen steady state at a selected finite region of array centering around the middle of the reaction domain. The development of spatial profile at different times is shown in Figs. 1(a)–1(f). It is apparent that irregularity tends to be erased out with time before asymptotically approaching an inhomogeneous state of sech form [The dotted line in Fig. 1(f) represents the fitting curve.] Our numerical simulation corroborates the analytic results.

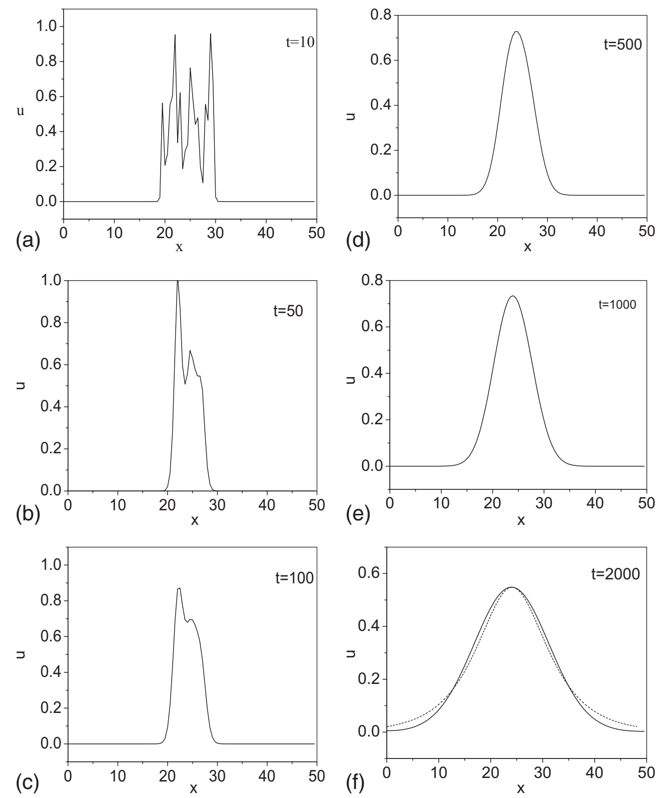

FIG. 1. [(a)–(f)] Development of inhomogeneity around a linearly stable state $u=0$ toward attaining the profile of sech form. Spatial profile at (a) $t=10$, (b) $t=50$, (c) $t=100$, (d) $t=500$, (e) $t=1000$, and (f) $t=2000$ (dotted line: fitting curve of sech form).

In this Brief Report we have shown that reaction can induce instability on a homogeneous stable state in a one-component reaction-diffusion system. The origin of this instability is the characteristic nonlinearity of the source term in contrast to the diffusion-induced Turing instability in a two-component system. The offshoot of this instability is a growth of an inhomogeneous distribution of sec-hyperbolic form reminiscent of a solitary wavelike structure. While Turing instability is based on the linear analysis of an infinitesimal perturbation and crucially depends on the characteristic length scale of the reaction medium, the reaction-induced solitary instability is a result of the nonlinear analysis of a finite spatiotemporal perturbation. Our numerical simulation is in good agreement with the analytical result. We hope that the instability can be realized in a systematically designed chemical reaction comprising of autocatalytic steps. A possible extension of the nonlinear theory to two-component systems outside the Turing space is worth pursuing for further exploration of reaction-induced instability of the homogeneous stable state.

Partial financial support from the Council of Scientific and Industrial Research, Government of India, is thankfully acknowledged.

[1] N. F. Britton, *Reaction-Diffusion Equations and their Applications to Biology* (Academic, New York, 1986).

[2] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers* (Birkhauser, Boston, 1997), Chap. 8.

[3] I. R. Epstein and J. A. Pojman, *An Introduction to Nonlinear Chemical Dynamics: Oscillations, Waves, Patterns and Chaos* (Oxford University Press, New York, 1998).

[4] A. Scott, *Nonlinear Science, Emergence and Dynamics of Coherent Structures* (Oxford University Press, New York, 2003).

[5] Ya. B. Zeldovich and D. A. Frank-Kamenetsky, Dokl. Akad. Nauk SSSR **19**, 693 (1938); Ya. B. Zeldovich and G. I. Darenblatt, Combust. Flame **3**, 61 (1959).

[6] A. L. Hodgkin and A. F. Huxley, J. Physiol. (London) **117**, 500 (1952); A. F. Huxley, *ibid.* **148**, 80 (1959); A. Scott, *Neu-*

roscience: A Mathematical Primer (Springer-Verlag, New York, 2002).
[7] R. FitzHugh, Biophys. J. **1**, 445 (1961); J. Nagumo, S. Arimoto, and S. Yoshizawa, Proc. IRE **50**, 2061 (1962).
[8] F. Offner, A. Weinberg, and C. Young, Bull. Math. Biophys. **2**, 89 (1940).
[9] P. de Kepper, I. R. Epstein, and K. Kautin, J. Am. Chem. Soc. **103**, 6121 (1981).
[10] A. Hanna, A. Saul, and K. Showalter, J. Am. Chem. Soc. **104**, 3838 (1982).
[11] J. D. Murray, *Mathematical Biology* (Springer-Verley, Berlin, 1993).
[12] See, for example, S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley, Reading, MA, 1994).
[13] U. Dressler and J. D. Farmer, Physica D **59**, 365 (1992).
[14] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, and A. Vulpiani, Phys. Rev. Lett. **77**, 1262 (1996).
[15] A. Torcini, P. Grassberger, and A. Politi, J. Phys. A **28**, 4533 (1995).
[16] P. S. Landa, A. A. Zaikin, and L. Schimansky-Geier, Chaos, Solitons Fractals **9**, 1367 (1998).
[17] C. Van den Broeck, J. M. R. Parrondo, J. Armero, and A. Hernandez-Machado, Phys. Rev. E **49**, 2639 (1994).
[18] F. Sagués, J. M. Sancho, and J. Garcia-Ojalvo, Rev. Mod. Phys. **79**, 829 (2007).
[19] S. Dutta, S. S. Riaz, and D. S. Ray, Phys. Rev. E **71**, 036216 (2005).
[20] S. S. Riaz, S. Dutta, S. Kar, and D. S. Ray, Eur. Phys. J. B **47**, 255 (2005).
[21] P. Ghosh, S. Sen, and D. S. Ray, Phys. Rev. E **79**, 016206 (2009).
[22] A. Bhattacharyay and J. K. Bhattacharjee, Eur. Phys. J. B **21**, 561 (2001).
[23] A. M. Turing, Philos. Trans. R. Soc. London, Ser. B **237**, 37 (1952).
[24] D. S. Ray, Phys. Lett. **102A**, 99 (1984).
[25] S. Dutta Gupta and D. S. Ray, Phys. Rev. B **40**, 10604 (1989).